

The effects due to hadronization in the inclusive τ lepton decay

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1 Introduction

The main subject of this paper is the role of effects due to hadronization in the theoretical description of inclusive τ lepton decay. This paper employs the results of the studies of this strong interaction process elaborated over past several years [1–6] and presents some new results recently obtained in this field [7].

The τ lepton is the only lepton which is heavy enough to decay into hadrons. This feature enables one to use this process in tests of Quantum Chromodynamics (QCD) and entire Standard Model. The theoretical description of inclusive τ lepton hadronic decay, similarly to the case of electron–positron annihilation into hadrons, requires no such phenomenological models as, for example, the so-called “Parton Distribution Functions” involved in the analysis of Deep Inelastic Scattering processes. It is worthwhile to note also that the experimental measurements of τ lepton decay are of a high accuracy. But the most interesting feature of this process is that it probes the hadron dynamics at energies below the mass of τ lepton.

The experimentally measurable quantity here is the ratio of the total width of τ lepton decay into hadrons to the width of its leptonic decay, which can be decomposed into three parts, namely

$$R_\tau = \frac{\Gamma(\tau^- \rightarrow \text{hadrons}^- \nu_\tau)}{\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau)} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}. \quad (1)$$

In the right-hand side of this equation, the first two terms account for the hadronic decay modes involving light quarks (u, d) only and associated with vector (V) and axial-vector (A) quark currents, respectively, whereas the last term accounts for the decay modes which involve strange quark. Each of the first two terms can be further decomposed into two parts according to the angular momentum in the hadronic rest frame, namely

$$R_{\tau,V} = R_{\tau,V}^{J=0} + R_{\tau,V}^{J=1}, \quad R_{\tau,A} = R_{\tau,A}^{J=0} + R_{\tau,A}^{J=1}. \quad (2)$$

In what follows we shall restrict ourselves to the consideration of parts $R_{\tau,V}^{J=1}$ and $R_{\tau,A}^{J=1}$ of ratio R_τ (1).

2 Theoretical description of τ lepton decay

The theoretical prediction for the quantities on hand (2) reads

$$R_{\tau, V/A}^{J=1} = \frac{N_c}{2} |V_{ud}|^2 S_{EW} (\Delta_{QCD}^{V/A} + \delta'_{EW}), \quad (3)$$

where $N_c = 3$ is the number of colors, $|V_{ud}| = 0.9738 \pm 0.0005$ is Cabibbo–Kobayashi–Maskawa matrix element [8], $S_{EW} = 1.0194 \pm 0.0050$ and $\delta'_{EW} = 0.0010$ stand for the electroweak corrections (see Refs. [9–11]), and

$$\Delta_{QCD}^{V/A} = 2 \int_{m_{V/A}^2}^{M_\tau^2} f\left(\frac{s}{M_\tau^2}\right) R^{V/A}(s) \frac{ds}{M_\tau^2} \quad (4)$$

denotes the QCD contribution. Here $M_\tau = 1.777 \text{ GeV}$ is the mass of τ lepton [8], $m_{V/A}$ stands for the total mass of the lightest allowed hadronic decay mode of τ lepton in the corresponding channel, $f(x) = (1-x)^2(1+2x)$, and

$$R^{V/A}(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} [\Pi^{V/A}(s + i\varepsilon) - \Pi^{V/A}(s - i\varepsilon)] = \frac{1}{\pi} \text{Im} \lim_{\varepsilon \rightarrow 0_+} \Pi^{V/A}(s + i\varepsilon), \quad (5)$$

with $\Pi^{V/A}(q^2)$ being the hadronic vacuum polarization function. In what follows the superscripts “V” and “A” will only be shown when relevant.

In general, it is convenient to perform the theoretical analysis of inclusive τ lepton decay in terms of the Adler function [12]

$$D(Q^2) = -\frac{d\Pi(-Q^2)}{d \ln Q^2}, \quad Q^2 = -q^2 = -s. \quad (6)$$

In the framework of perturbation theory its ultraviolet behavior can be approximated by power series in the strong running coupling $\alpha_s(Q^2)$

$$D(Q^2) \simeq D_{\text{pert}}^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j [\alpha_s^{(\ell)}(Q^2)]^j, \quad Q^2 \rightarrow \infty, \quad (7)$$

where at the one-loop level (i.e., for $\ell = 1$) $\alpha_s^{(1)}(Q^2) = 4\pi/(\beta_0 \ln z)$, $z = Q^2/\Lambda^2$, $\beta_0 = 11 - 2n_f/3$, Λ denotes the QCD scale parameter, n_f is the number of active flavors ($n_f = 2$ will be assumed hereinafter), and $d_1 = 1/\pi$, see papers [13–15] and references therein for the details. It is worth noting also that the function $R(s)$ (5) and the Adler function (6) can be expressed in terms of each other by making use of the following relations (see Refs. [12, 16, 17] for the details)

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta} \quad \longleftrightarrow \quad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2} ds. \quad (8)$$

In first of these equations the integration contour in the complex ζ -plane lies in the region of analyticity of the integrand.

It is important to outline here that all the mentioned above is only valid for “true physical” hadronic vacuum polarization function $\Pi_{\text{phys}}(q^2)$ and Adler function $D_{\text{phys}}(Q^2)$. However, as it often happens, one has to deal with their perturbative approximations $\Pi_{\text{pert}}(q^2)$ and $D_{\text{pert}}(Q^2)$, which are valid in the ultraviolet asymptotic only. Besides, expressions $\Pi_{\text{pert}}(q^2)$ and $D_{\text{pert}}(Q^2)$ are inconsistent with dispersion relation (8), which is determined by the kinematics of physical process on hand.

Thus, one arrives at the point where the results of perturbation theory need to be “merged” with relevant dispersion relations. This objective can be achieved in the framework of “Dispersive approach” to QCD, which will be briefly overviewed in Sect. 4. The theoretical description of inclusive τ lepton hadronic decay within Dispersive approach will be performed in Sect. 4, whereas the analysis of this process within perturbative approach will be discussed in Sect. 3.

3 Perturbative approach

In this Section, we shall study the massless limit, that implies that the masses of all final state particles are neglected. In this case, by making use of definitions (5) and (6), integrating by parts, and employing Cauchy theorem, the quantity Δ_{QCD} (4) can be represented as

$$\Delta_{\text{QCD}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(M_{\tau}^2 e^{i\theta}) (1 + 2e^{i\theta} - 2e^{i3\theta} - e^{i4\theta}) d\theta, \quad (9)$$

see, e.g., papers [9, 18–20] and references therein. In general, in Eqs. (5) and (6) it is convenient to handle the leading contributions (i.e., the terms of 0-th order in the strong running coupling) separately from the contributions due to strong interaction, namely

$$R(s) = r^{(0)}(s) + r^{(\ell)}(s), \quad D(Q^2) = d^{(0)}(Q^2) + d^{(\ell)}(Q^2). \quad (10)$$

In what follows we shall restrict ourselves to the one-loop level ($\ell = 1$).

In fact, the only available option within perturbative approach is to directly use in the theoretical expression for Δ_{QCD} (despite of remarks given in Sect. 2) the perturbative approximation of hadronic vacuum polarization function $\Pi_{\text{pert}}(q^2)$ instead of its unknown “true physical” expression $\Pi_{\text{phys}}(q^2)$. For the case of functions (9) and (7), this prescription eventually results in (see Ref. [7] for the details)

$$r^{(0)}(s) = 1 \quad \longleftrightarrow \quad d^{(0)}(Q^2) = 1, \quad (11)$$

$$\Delta_{\text{pert}} = 1 + \frac{4}{\beta_0} \int_0^{\pi} \frac{c_0 A_1(\theta) + \theta A_2(\theta)}{\pi(c_0^2 + \theta^2)} d\theta, \quad (12)$$

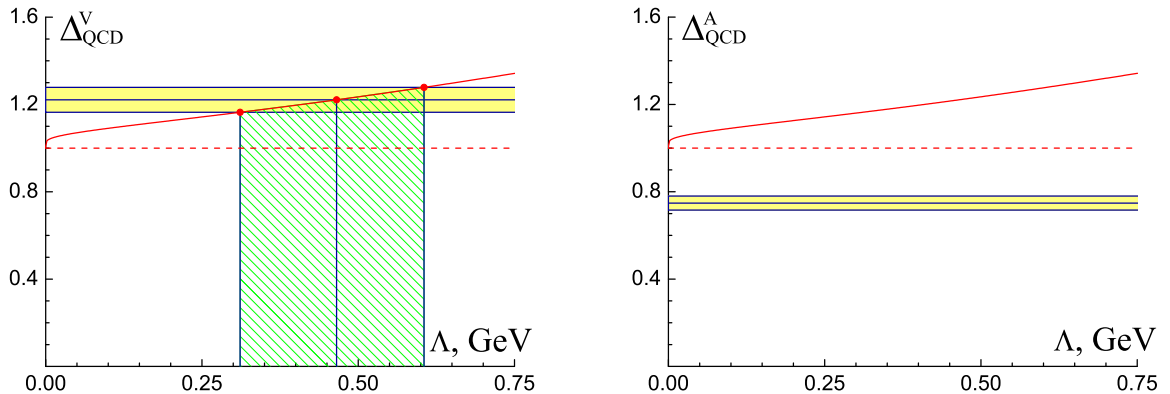


Figure 1: Comparison of the one-loop perturbative expression Δ_{pert} (12) (solid curves) with relevant experimental data (14) (horizontal shaded bands). The leading-order terms of Δ_{pert} (12) are denoted by horizontal dashed lines. The solution for QCD scale parameter Λ (if exists) is shown by vertical dashed band. The left and right plots correspond to vector and axial-vector channels, respectively.

where $c_0 = \ln(M_\tau^2/\Lambda^2)$ and

$$A_1(\theta) = 1 + 2\cos(\theta) - 2\cos(3\theta) - \cos(4\theta), \quad A_2(\theta) = 2\sin(\theta) - 2\sin(3\theta) - \sin(4\theta). \quad (13)$$

Let us proceed now to the comparison of one-loop perturbative result (12) with corresponding experimental data. First of all, it is worth emphasizing here that perturbative approach gives identical predictions for functions $\Delta_{\text{QCD}}^{V/A}$ (4) in vector and axial-vector channels (i.e., $\Delta_{\text{pert}}^V \equiv \Delta_{\text{pert}}^A$). However, their experimental values extracted from data presented in Refs. [20–22] are different, namely

$$\Delta_{\text{exp}}^V = 1.221 \pm 0.057, \quad \Delta_{\text{exp}}^A = 0.748 \pm 0.032. \quad (14)$$

These quantities are juxtaposed with perturbative result (12) in Fig. 1. As one can infer from this figure, for vector channel the corresponding value of QCD scale parameter is $\Lambda = (465_{-154}^{+140})$ MeV (formally, there is also the second solution, $\Lambda = (1646_{-29}^{+26})$ MeV, which will not be considered hereinafter). As for the axial-vector channel, the perturbative approach fails to describe experimental data on τ lepton hadronic decay, since for any value of Λ the function Δ_{pert} (12) exceeds Δ_{exp}^A (14).

4 Dispersive approach

4.1 General remarks

It is crucial to emphasize that the analysis presented in Sect. 3 entirely leaves out the effects due to hadronization, which play an important role in the studies of strong

interaction processes at low energies. Specifically, the mathematical realization of the physical fact, that in a strong interaction process no final state hadrons can be produced at energies below the total mass of the lightest allowed hadronic final state, consists in the fact that the beginning of cut of corresponding hadronic vacuum polarization function $\Pi(q^2)$ in complex q^2 -plane is located at the threshold of hadronic production, but not at the point $q^2 = 0$. Such limitations are inherently embodied within relevant dispersion relations, which, in turn, impose stringent physical nonperturbative constraints on the quantities on hand. Obviously, these restrictions should certainly be accounted for when one is trying to go beyond the limits of perturbation theory.

The nonperturbative constraints, which dispersion relation (8) imposes on the Adler function (6), have been merged with perturbative result (7) in the framework of Dispersive approach to QCD, that has eventually led to the following integral representations for functions (5) and (6) (see Refs. [3, 4, 6] for the details):

$$R(s) = r^{(0)}(s) + \theta\left(1 - \frac{m^2}{s}\right) \int_s^\infty \rho(\sigma) \frac{d\sigma}{\sigma}, \quad (15)$$

$$D(Q^2) = d^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^\infty \rho(\sigma) \frac{\sigma - m^2}{\sigma + Q^2} \frac{d\sigma}{\sigma}, \quad (16)$$

where $\theta(x)$ is the unit step-function ($\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise) and $\rho(\sigma)$ denotes the so-called spectral density. It is worth mentioning that in the massless limit ($m \rightarrow 0$) expressions (15) and (16) become identical to those of the so-called “Analytic perturbation theory” [23–26]. But it is essential to keep the hadronic mass m nonvanishing within the approach on hand.

Let us proceed now to the description of inclusive τ lepton hadronic decay within Dispersive approach. It is worthwhile to note here that there are two distinctions between the approach on hand and the massless perturbative approach presented in Sect. 3. Specifically, the first distinction is the incorporation of effects due to hadronization, and the second one is the expression for the one-loop spectral density

$$\rho(\sigma) = \frac{4}{\beta_0} \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2} + \frac{\Lambda^2}{\sigma}, \quad (17)$$

which resembles previously studied nonperturbative model [1, 2, 6, 27–29]. In the right-hand side of Eq. (17) the first term is the one-loop perturbative contribution whereas the second term represents intrinsically nonperturbative part of the spectral density.

4.2 Abrupt kinematic threshold

There are two options in the framework of Dispersive approach. The first one is the so-called “abrupt kinematic threshold”, which implies that the leading-order term

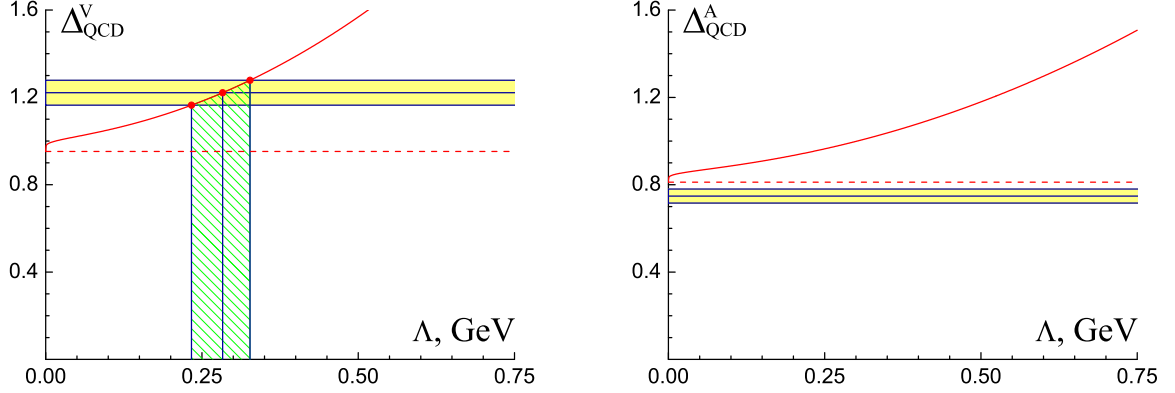


Figure 2: Comparison of expression (19) (solid curves) with relevant experimental data (14) (horizontal shaded bands). The leading-order terms of $\Delta_{\text{QCD}}^{\text{V/A}}$ (19) are denoted by horizontal dashed lines. The solution for QCD scale parameter Λ (if exists) is shown by vertical dashed band. The left and right plots correspond to vector and axial-vector channels, respectively.

of $R(s)$ (5) is approximated by the step-function

$$r_{\text{V/A}}^{(0)}(s) = \theta\left(1 - \frac{m_{\text{V/A}}^2}{s}\right) \longleftrightarrow d_{\text{V/A}}^{(0)}(Q^2) = \frac{Q^2}{Q^2 + m_{\text{V/A}}^2}, \quad (18)$$

see papers [3, 4, 6] and references therein for the details. Equation (18) accounts only for basic kinematic restriction on hadronic vacuum polarization function $\Pi(q^2)$ and, in fact, represents a rather rough approximation. In this case the quantity $\Delta_{\text{QCD}}^{\text{V/A}}$ (4) reads [6]

$$\Delta_{\text{QCD}}^{\text{V/A}} = 1 - g(\zeta_{\text{V/A}}) + \int_{m_{\text{V/A}}^2}^{\infty} H\left(\frac{\sigma}{M_\tau^2}\right) \rho(\sigma) \frac{d\sigma}{\sigma}, \quad (19)$$

where $\zeta_{\text{V/A}} = m_{\text{V/A}}^2/M_\tau^2$ and

$$H(x) = g(x) \theta(1 - x) + g(1) \theta(x - 1) - g(\zeta_{\text{V/A}}), \quad g(x) = x(2 - 2x^2 + x^3). \quad (20)$$

However, similarly to perturbative case (12), expression (19) is unable to describe the experimental data on τ lepton hadronic decay in axial-vector channel (14), see Fig. 2.

4.3 Smooth kinematic threshold

The second, more accurate, option within Dispersive approach is the so-called “smooth kinematic threshold”. In this case the leading-order term of $R(s)$ (5) takes the following form [7]

$$r_{\text{V/A}}^{(0)}(s) = \left(1 - \frac{m_{\text{V/A}}^2}{s}\right)^{3/2} \longleftrightarrow d_{\text{V/A}}^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left\{ 1 + u(\xi) \ln \sqrt{1 + 2\xi[1 - u(\xi)]} \right\}, \quad (21)$$

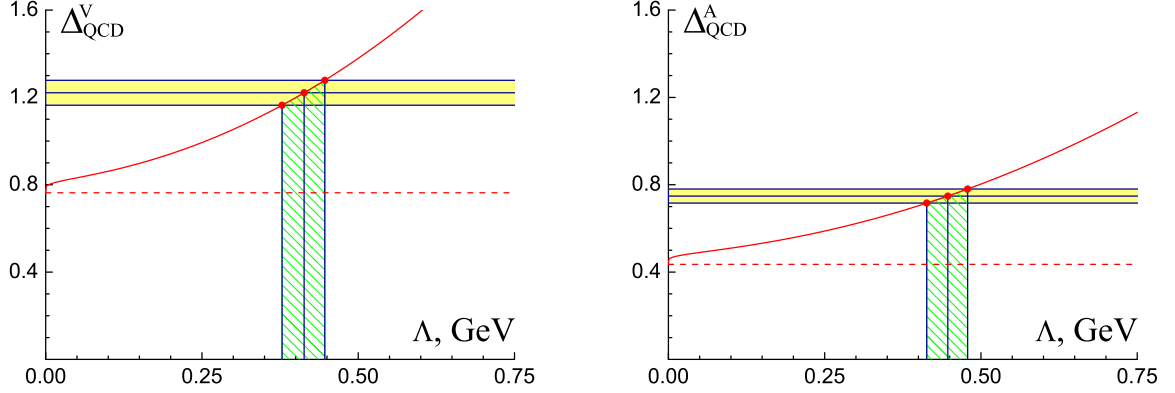


Figure 3: Comparison of expression (22) (solid curves) with relevant experimental data (14) (horizontal shaded bands). The leading-order terms of $\Delta_{\text{QCD}}^{\text{V/A}}$ (22) are denoted by horizontal dashed lines. The solutions for QCD scale parameter Λ are shown by vertical dashed bands. The left and right plots correspond to vector and axial-vector channels, respectively.

where $u(\xi) = \sqrt{1 + \xi^{-1}}$ and $\xi = Q^2/m_{\text{V/A}}^2$. Eventually this leads to the following expression for the quantity $\Delta_{\text{QCD}}^{\text{V/A}}$ (4):

$$\begin{aligned} \Delta_{\text{QCD}}^{\text{V/A}} = & \sqrt{1 - \zeta_{\text{V/A}}} \left(1 + 6\zeta_{\text{V/A}} - \frac{5}{8}\zeta_{\text{V/A}}^2 + \frac{3}{16}\zeta_{\text{V/A}}^3 \right) + \int_{m_{\text{V/A}}^2}^{\infty} H\left(\frac{\sigma}{M_\tau^2}\right) \rho(\sigma) \frac{d\sigma}{\sigma} \\ & - 3\zeta_{\text{V/A}} \left(1 + \frac{1}{8}\zeta_{\text{V/A}}^2 - \frac{1}{32}\zeta_{\text{V/A}}^3 \right) \ln \left[\frac{2}{\zeta_{\text{V/A}}} (1 + \sqrt{1 - \zeta_{\text{V/A}}}) - 1 \right], \end{aligned} \quad (22)$$

see paper [7] and references therein for the details.

It is worth noting also that in the massless limit ($m \rightarrow 0$) both equations (19) and (22) acquire the same form

$$\Delta_{\text{QCD}} = 1 + \int_0^\infty h\left(\frac{\sigma}{M_\tau^2}\right) \rho(\sigma) \frac{d\sigma}{\sigma}, \quad h(x) = g(x) \theta(1 - x) + g(1) \theta(x - 1). \quad (23)$$

In the perturbative case the difference between expressions (23) and (12) is due to the residue term

$$\Delta_{\text{res}} = \frac{4}{\beta_0} h_1\left(\frac{\Lambda^2}{M_\tau^2}\right), \quad h_1(x) = h_2(x) \theta(1 - x) + h_2(1) \theta(x - 1), \quad h_2(x) = x(2 - 2x^2 - x^3), \quad (24)$$

which appears to be additionally accounted for in Eq. (12), see discussion of this issue in paper [7].

The comparison of obtained result (22) with experimental data (14) yields nearly identical solutions for QCD scale parameter Λ in both channels, see Fig. 3. Namely,

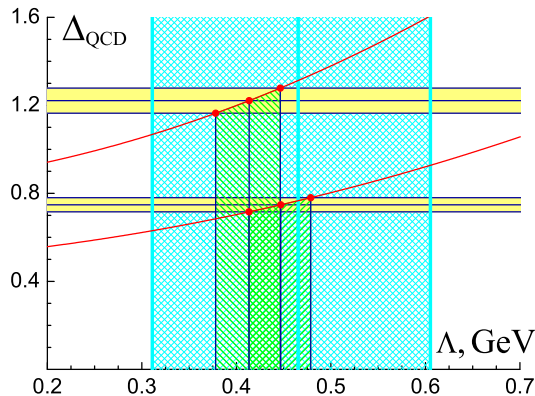


Figure 4: The solutions for QCD scale parameter Λ obtained within Dispersive approach (22) for vector and axial-vector channels (vertical dashed green bands) and perturbative solution (12) corresponding to vector channel (vertical dashed light-blue band).

$\Lambda = (412 \pm 34)$ MeV for vector channel and $\Lambda = (446 \pm 33)$ MeV for axial-vector one. Besides, as one can infer from Fig. 4, both these solutions for QCD scale parameter agree very well with perturbative solution $\Lambda = (465^{+140}_{-154})$ MeV obtained in Sect. 3.

5 Conclusions

Theoretical description of inclusive τ lepton hadronic decay is performed in the framework of Dispersive approach to QCD. The significance of effects due to hadronization is convincingly demonstrated. The approach on hand is capable of describing experimental data on τ lepton decay in vector and axial-vector channels. The vicinity of values of QCD scale parameter Λ obtained in both channels testifies to the self-consistency of developed approach.

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